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A slippage test for a set of Gamma-variates

by

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1. Summary

In this report a generalization is given of the significance tests for the largest and the smallest respectively of a set of estimated normal variances as suggested by W.G. COCHRAN (1941) and one of the present authors (cf. R. Doornbos (1956)) respectively. These tests only deal with the case where the samples from which the variances are estimated all have the same size.

The present report gives a treatment which is also valid for different sample sizes. Further we consider the power function of the tests with respect to the alternative hypothesis that one of the variances has slipped to the right or, in the case of the test for the smallest variances to the left. Slippage tests for a set of Poisson-variates which appear to lead to the same distribution-functions will be discussed in a separate report by the second author.

Finally the construction of a nomogram is suggested to faciliate the application of the tests.

2. Introduction and description of the tests

Suppose we have a set of random variables

(2.1) $$\underline{u}_1, \dots, \underline{u}_k \qquad ^{1)}$$

distributed independently of one another according to gamma distributions with parameters $\alpha_1, \beta_1; \dots; \alpha_k, \beta_k$ respectively; that is to say the density function of $\underline{u}_i$ is

(2.2) $$f(u_i) = \frac{1}{\Gamma(\alpha_i)\beta_i^{\alpha_i}} u_i^{\alpha_i - 1} e^{-u_i/\beta_i}, \quad 0 \leqq u \leqq \infty,$$

where α_i and β_i are real positive numbers. As is well known the distribution of $\underline{t} = \underline{\chi}^2 \cdot \sigma^{-2}$, where $\underline{\chi}^2$ is distributed as a chi-square with ν degrees of freedom, is a special case of a gamma distribution, with parameters $\alpha = \nu/2$ and $\beta = 2\sigma^{-2}$.

Now our problem is to find tests for the hypothesis

(2.3) $$H_0 : \beta_1 = \dots = \beta_k = \beta, \text{ say},$$

against the alternatives

(2.4) $$H_1 : \beta_1 = \dots = \beta_{i-1} = \beta_{i+1} = \dots = \beta_k = \beta,$$
$$\beta_i = c_{1i}\beta, \quad c_{1i} > 1,$$

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1) Random variables are denoted by underlined symbols.

for one unknown value of i

and

(2.5) $$H_2: \beta_1 = \dots = \beta_{i-1} = \beta_{i+1} = \dots = \beta_k = \beta, \quad \beta_i = c_{2i}\beta, \quad 0 < c_{2i} < 1,$$

for one unknown value of i.

For both tests we compute the ratios

(2.6) $$x_j = \frac{u_j}{\Sigma u_i}, \quad (j = 1, \dots, k).$$

Then, if we are testing H_0 against H_1, the following incomplete B integrals are determined:

(2.7) $$d_j = \frac{1}{B(\alpha_j, A-\alpha_j)} \int_{x_j}^{1} x^{\alpha_j-1}(1-x)^{A-\alpha_j-1}\,dx = 1 - I_{x_j}(\alpha_j, A-\alpha_j), \quad (j = 1, \dots, k),$$

where $A = \sum_1^k \alpha_i$. Next we define the test statistic d by

(2.8) $$d = \min d_j$$

If we reject H_0 when d takes a value $d \leqq \frac{\varepsilon}{k}$, the level of significance lies between ε and $\varepsilon - \frac{1}{2}\varepsilon^2$ as will be shown in section 4.

Testing H_0 against H_2 requires computation of the integrals

(2.9) $$e_j = \frac{1}{B(\alpha_j, A-\alpha_j)} \int_0^{x_j} x^{\alpha_j-1}(1-x)^{A-\alpha_j-1}\,dx = 1 - d_j = I_{x_j}(\alpha_j, A-\alpha_j).$$

We reject H_0 if

(2.10) $$e = \min e_j \leqq \frac{\varepsilon}{k}.$$

The level of significance is again a number between ε and $\varepsilon - \frac{1}{2}\varepsilon^2$.

3. An optimum property of the tests if $\alpha_1 = \dots = \alpha_k$

D.R. TRUAX (1953) proved an optimum property of COCHRAN's test. In exactly the same way one can prove that our tests are optimal in the following sence if $\alpha_1 = \dots = \alpha_k$. Let D_0 be the decision that H_0 is true and let D_{1j} be the decision that H_0 is false and that $\beta_j = \max(\beta_1, \dots, \beta_k)$. Then, if $d = d_m$, i.e. if d_m is the smallest of $d_1, \dots, d_k$ the procedure

(3.1) $$\begin{aligned} &\text{if } d \leqq L_\varepsilon \quad \text{select } D_{1m}, \\ &\text{if } d > L_\varepsilon \quad \text{select } D_0, \end{aligned}$$

where L_ε is a constant determined by the condition that D_o should be selected with probability $1-\varepsilon$ if H_0 is true, maximizes the probability of making the correct decision if the hypothesis H_1 is true.

When the hypothesis H_2 is true the analogous optimum property holds for our second test. In both cases $\frac{\varepsilon}{k}$ is an approximation of the critical values of $\underline{d}$ and $\underline{e}$.

4. Proofs of the results stated in 2.

To obtain the joint distribution of $\underline{x}_1, \ldots, \underline{x}_{k-1}$ as given by (2.6) and of $\underline{U} = \underline{u}_1 + \cdots + \underline{u}_k$ we put

$$(4.1) \qquad \begin{cases} u_1 = x_1 U, \\ \vdots \\ u_{k-1} = x_{k-1} U, \\ u_k = U(1 - x_1 - \cdots - x_k). \end{cases}$$

The Jacobian of this transformation becomes

$$(4.2) \qquad \begin{vmatrix} U & 0 & \cdots & 0 & x_1 \\ 0 & U & \cdots & 0 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & U & x_{k-1} \\ -U & -U & \cdots & -U & 1-x_1-\cdots-x_{k-1} \end{vmatrix} = \begin{vmatrix} U & 0 & \cdots & 0 & x_1 \\ 0 & U & \cdots & 0 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & U & x_{k-1} \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = U^{k-1}.$$

Under H_0 the simultaneous distribution of $\underline{u}_1, \ldots, \underline{u}_k$ is (2.2)

$$(4.3) \qquad f(u_1, \ldots, u_k) = \frac{u_1^{\alpha_1 - 1} \cdots u_k^{\alpha_k - 1} e^{-(u_1 + \cdots + u_k)/\beta}}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)\,\beta^A},$$

where $A = \alpha_1 + \cdots + \alpha_k$.

Thus the joint distribution of $\underline{x}_1, \ldots, \underline{x}_{k-1}$ and $\underline{U}$ is given by the density function

$$(4.4) \qquad \begin{aligned} g(x_1, \ldots, x_{k-1}, U) &= \frac{x_1^{\alpha_1 - 1} \cdots x_{k-1}^{\alpha_{k-1}-1} (1 - x_1 - \cdots - x_{k-1})^{\alpha_k - 1} U^{A-1} e^{-U/\beta}}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)\,\beta^A} = \\ &= \frac{\Gamma(A)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)} x_1^{\alpha_1 - 1} \cdots x_{k-1}^{\alpha_{k-1}-1} (1 - x_1 - \cdots - x_{k-1})^{\alpha_k - 1} \frac{U^{A-1} e^{-U/\beta}}{\Gamma(A)\,\beta^A} \end{aligned}$$

Thus we see, as is well known that u has also a gamma distribution, with parameters $\alpha_1 + \cdots + \alpha_k = A$ and β and moreover that the joint distribution of $x_1, \ldots, x_{k-1}$ is given by

$$(4.5) \qquad h(x_1, \ldots, x_{k-1}) = \frac{\Gamma(A)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_k)} x_1^{\alpha_1-1} \ldots x_{k-1}^{\alpha_{k-1}-1}(1-x_1 \cdots - x_{k-1})^{\alpha_k - 1}.$$

But the same derivation gives us the general result

$$(4.6) \qquad p(x_1, \ldots, x_i) = \frac{\Gamma(A)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_i)\Gamma(\alpha_{i+1}+\cdots+\alpha_k)} x_1^{\alpha_1-1} \cdots x_i^{\alpha_i-1}(1-x_1 \ldots - x_i)^{\alpha_{i+1}+\cdots+\alpha_k-1},$$
$$(i = 1, \ldots, k-1),$$

if we consider instead of $u_1, \ldots, u_k$ the $i+1$ variables $u_1, \ldots, u_i$ and $u_{i+1} + \cdots + u_k$ which are also independent from one another and which have gamma distributions with parameters $\alpha_1, \ldots, \alpha_i$, $\alpha_{i+1} + \cdots + \alpha_k$ and β.

We consider now a set of k real numbers $g_1, \ldots, g_k$ $(0 \leq g_i \leq 1)$ and the probabilities defined by

$$(4.7) \qquad \begin{cases} p_i = P[x_i \leq g_i] \\ p_{ij} = P[x_i \leq g_i \text{ and } x_j \leq g_j], \ (i \neq j) \\ \ldots \\ q_i = P[x_i > g_i] \\ q_{ij} = P[x_i > g_i \text{ and } x_j > g_j], \ (i \neq j) \\ \ldots \end{cases}$$

all computed under H_0. If we denote by P the probability that at least one of the ratios x_i does not exceed the corresponding value g_i, we have

$$(4.8) \qquad P = \sum p_i - \sum p_{ij} + \sum p_{ij,l} \cdots + (-1)^{k-1} p_{1,2,\ldots,k}$$

where the r th summation is extended over all p's with r subscripts; hence the r th sum has $\binom{k}{r}$ terms (G.W. FELLER (1950), Chapter 4).

For Q, the probability that at least one of the x_i exceeds g_i, we have

$$(4.9) \qquad Q = \sum q_i - \sum q_{ij} + \sum q_{ij,l} + \cdots + (-1)^{k-1} q_{1,2,\ldots,k}.$$

It follows from BONFERRONI's inequality (cf. W. FELLER (1950), chapter 4) that

(4.10) $$\sum p_i - \sum p_{i,j} \leqq P \leqq \sum p_i$$

and

(4.11) $$\sum q_i - \sum q_{i,j} \leqq Q \leqq \sum q_i .$$

In the following we shall prove the inequalities

(4.12) $$p_{i,j} \leqq p_i p_j$$

and

(4.13) $$q_{i,j} \leqq q_i q_j$$

If we now determine the numbers $g^{(2)}_{i,\varepsilon}$ so that all p_i are equal to $\frac{\varepsilon}{k}$, then we get from (4.10) and (4.12)

$$\sum p_i - \sum p_i p_j \leq \sum p_i - \sum p_{i,j} \leq P_\varepsilon \leq \sum p_i ,$$

or

$$\varepsilon - \frac{1}{2} \frac{k-1}{k} \varepsilon^2 \leq P_\varepsilon \leq \varepsilon ,$$

or for $k \geqq 2$

(4.14) $$\varepsilon - \frac{1}{2} \varepsilon^2 \leq P_\varepsilon \leq \varepsilon$$

In the same way we get

(4.15) $$\varepsilon - \frac{1}{2} \varepsilon^2 \leqq Q_\varepsilon \leq \varepsilon$$

if the numbers $g^{(1)}_{i,\varepsilon}$ are determined so as to make all q_i equal to $\frac{\varepsilon}{k}$. As the procedure described in section 2 to test H_0 against the sets of alternatives H_2 and H_1 respectively gives us the probabilities P_ε and Q_ε respectively of rejecting H_0 when H_0 is true, these probabilities lie between the bounds stated there.

We now proceed to prove the inequalities (4.12) and (4.13). First it is shown that (4.12) and (4.13) are equivalent. We have

$$p_i = 1 - q_i \text{ and } p_j = 1 - q_j$$

and consequently

(4.16) $$p_i (1-p_j) = q_j (1-q_i)$$

Further

(4.17) $$p_i - p_{i,j} = q_j - q_{i,j} \; (= P[x_i \leq g_i \text{ and } x_j > g_j])$$

From (4.16) and (4.17) we obtain

(4.18) $\quad p_i p_j - p_{i,j} = q_i q_j - q_{i,j}$,

which proves the equivalence of (4.12) and (4.13). Thus it is sufficient to prove (4.12) [1] and we need only consider values q_i and q_j such that $q_i + q_j \leqq 1$, for when $q_i + q_j > 1$, $q_{i,j} = 0$ and so (4.13) and (4.12) are obviously true.

It is easily seen that (4.12) is equivalent with

(4.19) $$\frac{p_{i,j}}{p_j} \leqq \frac{p_i - p_{i,j}}{q_j},$$

or

(4.20) $$\frac{P[x_i \leqq g_i \text{ and } x_j \leqq g_j]}{P[x_j \leqq g_j]} \leqq \frac{P[x_i \leqq g_i \text{ and } x_j > g_j]}{P[x_j > g_j]}.$$

Fron (4.6) it follows that the left hand member $L(g_i, g_j)$ of (4.20) equals

$$C \cdot \frac{\int_0^{g_j}\int_0^{g_i} x_j^{\alpha_j - 1} x_i^{\alpha_i - 1}(1 - x_i - x_j)^{A-\alpha_i-\alpha_j-1}\, dx_i\, dx_j}{\int_0^{g_j} x_j^{\alpha_j-1}(1-x_j)^{A-\alpha_j-1}\, dx_j}$$

where $C = \dfrac{\Gamma(A-\alpha_j)}{\Gamma(\alpha_i)\Gamma(A-\alpha_i-\alpha_j)}$

Putting $x_i = v(1-x_j)$ we get

(4.21) $$L(g_i, g_j) = C \frac{\int_0^{g_j}\int_0^{\frac{g_i}{1-x_j}} v^{\alpha_i-1}(1-v)^{A-\alpha_i-\alpha_j-1} x_j^{\alpha_j-1}(1-x_j)^{A-\alpha_j-1}\, dv\, dx_j}{\int_0^{g_j} x_j^{\alpha_j-1}(1-x_j)^{A-\alpha_j-1}\, dx_j}$$

$$\leqq C \int_0^{\frac{g_i}{1-g_j}} v^{\alpha_i-1}(1-v)^{A-\alpha_i-\alpha_j-1}\, dv.$$

On the other hand the right hand member $R(g_i, g_j)$ of (4.20) is equal to

(4.22) $$C \cdot \frac{\int_{g_j}^1 \int_0^{\min(g_i, 1-x_j)} x_i^{\alpha_i-1} x_j^{\alpha_j-1}(1-x_i-x_j)^{A-\alpha_i-\alpha_j-1}\, dx_i\, dx_j}{\int_{g_j}^1 x_j^{\alpha_j-1}(1-x_j)^{A-\alpha_j-1}}$$

$$= C \frac{\int_{g_j}^1 \int_0^{\min\left(\frac{g_i}{1-x_j}, 1\right)} v^{\alpha_i-1}(1-v)^{A-\alpha_i-\alpha_j-1} x_j^{\alpha_j-1}(1-x_j)^{A-\alpha_j-1}\, dv\, dx_j}{\int_{g_j}^1 x_j^{\alpha_j-1}(1-x_j)^{A-\alpha_j-1}\, dx_j}$$

$$\geqq C \int_0^{\frac{g_i}{1-g_j}} v^{\alpha_i-1}(1-v)^{A-\alpha_i-\alpha_j-1}\, dv.$$

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1) The following proof, which is substantially simpler than another one which was developed by the authors, has been found by H. KESTEN, assistant of the Statistical Department,oas a specia case of the proof of the more general inequality

$$p_{i,j,\dots,l} \leqq p_i \cdot p_j \cdots p_l.$$

So it follows from (4.21) and (4.22) that (4.20) holds.

5. The power of the tests

In this section we shall derive upper and lawer bounds for the probabilities of making a correct decision, following the procedure described in section 3, under the hypotheses H_1 and H_2.

In the first case, i.e. when H_1 is true, we assume that β_i is the parameter which has slipped to the right. i.e. $\beta_i = c_{1i}\beta$, $c_{1i} > 1$. Then we prove that Q_i, the probability of making the correct decision lies between the limits

$$(5.1)\quad \left[1 - I_{c_i^{(1)}}(\alpha_i, A-\alpha_i)\right](1-\varepsilon) \leq Q_i \leq \left[1 - I_{c_i^{(1)}}(\alpha_i, A-\alpha_i)\right],$$

where

$$(5.2)\qquad c_i^{(1)} = \frac{g_{i,\varepsilon}^{(1)}}{c_{1i} - (c_{1i}-1)g_{i,\varepsilon}^{(1)}},$$

where $g_{i,\varepsilon}^{(1)}$ is determined so as to make

$$(5.3)\qquad I_{g_{i,\varepsilon}^{(1)}}(\alpha_i, A-\alpha_i) = 1 - \frac{\varepsilon}{k}$$

When c_{1i} becomes large Q_i converges to the upper bound given by the right hand member of (5.1).

When H_2 is true and β_j has slipped to the left, i.e. $\beta_j = c_{2j}\beta$, $0 \leq c_{2j} < 1$, the following limits can be derived for P_j, the probability of making the correct decision in this case.

$$(5.4)\quad \left[I_{c_j^{(2)}}(\alpha_j, A-\alpha_j)\right](1-\varepsilon) \leq P_j \leq I_{c_j^{(2)}}(\alpha_j, A-\alpha_j),$$

where

$$(5.5)\qquad c_j^{(2)} = \frac{g_{j,\varepsilon}^{(2)}}{c_{2j} + (1-c_{2j})g_{j,\varepsilon}^{(2)}}$$

and $g_{j,\varepsilon}^{(2)}$ is determined from

$$(5.6)\qquad I_{g_{j,\varepsilon}^{(2)}}(\alpha_j, A-\alpha_j) = \frac{\varepsilon}{k}.$$

Again for small values of c_{2j},

$$P_j \approx I_{c_j^{(2)}}(\alpha_j, A-\alpha_j).$$

In order to prove (5.1) we may assume without loss of generality that $i=1$ and then we put $u_1/c_{11} = v_1$, thus v_1 has a gamma distribution with parameters α_1 and β. The probability Q_1 of making the correct decision is

$$Q_1 = P\left[d_1 = \min_{j=1,\dots,k} d_j \text{ and } d_1 \leqq \frac{\varepsilon}{k}\right] \geqq$$
$$\geqq P\left[d_1 \leqq \frac{\varepsilon}{k} \text{ and } d_2 > \frac{\varepsilon}{k} \dots \text{ and } d_k > \frac{\varepsilon}{k}\right] =$$
$$= P\left[d_1 \leqq \frac{\varepsilon}{k}\right] - P\left[\left(d_1 \leqq \frac{\varepsilon}{k} \text{ and } d_2 \leqq \frac{\varepsilon}{k}\right); \text{or} \dots; \text{or} \left(d_1 \leqq \frac{\varepsilon}{k} \text{ and } d_k \leqq \frac{\varepsilon}{k}\right)\right].$$

Thus the following inequality holds

$$(5.7) \quad P\left[d_1 \leqq \frac{\varepsilon}{k}\right] - \sum_{2}^{k} P\left[d_1 \leqq \frac{\varepsilon}{k} \text{ and } d_j \leqq \frac{\varepsilon}{k}\right] \leqq Q_1 \leqq P\left[d_1 \leqq \frac{\varepsilon}{k}\right]$$

We have

$$P\left[d_1 \leqq \frac{\varepsilon}{k}\right] = P\left[x_1 \geqq g_{1,\varepsilon}^{(1)}\right] =$$
$$= P\left[\frac{c_{11} v_1}{v_1 + u_2 + \dots + u_k + (c_{11}-1) v_1} \geqq g_{1,\varepsilon}^{(1)}\right] =$$
$$= P\left[\frac{v_1}{v_1 + u_2 + \dots + u_k} \geqq \frac{g_{1,\varepsilon}^{(1)}}{c_{11} - (c_{11}-1) g_{1,\varepsilon}^{(1)}}\right] =$$
$$\text{(cf. (5.2))} \quad = P\left[\frac{v_1}{v_1 + u_2 + \dots + u_k} \geqq c_1^{(1)}\right].$$

The distribution of $\dfrac{v_1}{v_1 + u_2 + \dots + u_k}$ is the distribution of x_1 under H_0 and therefore known. In fact

$$(5.8) \quad P\left[d_1 \leqq \frac{\varepsilon}{k}\right] = 1 - I_{c_1^{(1)}}(\alpha_1, A - \alpha_1).$$

Further we have

$$(5.9) \quad P\left[d_1 \leqq \frac{\varepsilon}{k} \text{ and } d_j \leqq \frac{\varepsilon}{k}\right] =$$
$$= P\left[\frac{v_1}{v_1 + u_2 + \dots + u_k} \geqq c_1^{(1)} \text{ and } \frac{u_j}{v_1 + \dots + u_k + (c_{11}-1) v_1} \geqq g_{j,\varepsilon}^{(1)}\right] \leqq$$
$$\leqq P\left[\frac{v_1}{v_1 + u_2 + \dots + u_k} \geqq c_1^{(1)} \text{ and } \frac{u_j}{v_1 + \dots + u_k} \geqq g_{j,\varepsilon}^{(1)}\right] \leqq$$
(according to (4.13))
$$\leqq P\left[\frac{v_1}{v_1 + \dots + u_k} \geqq c_1^{(1)}\right] \cdot P\left[\frac{u_j}{v_1 + \dots + u_k} \geqq g_{j,\varepsilon}^{(1)}\right] =$$

$$= \left[1 - I_{c_1^{(1)}}(\alpha_1, A - \alpha_1)\right] \frac{\varepsilon}{k}.$$

Substituting (5.8) and (5.9) into (5.7) we get

$$(5.10)\quad \left[1 - I_{c_1^{(1)}}(\alpha_1, A-\alpha_1)\right](1-\varepsilon) \leq \left[1 - I_{c_1^{(1)}}(\alpha_1, A-\alpha_1)\right]\left(1 - \tfrac{k-1}{k}\varepsilon\right) \leq$$
$$\leq Q_1 \leq \left[1 - I_{c_1^{(1)}}(\alpha_1, A-\alpha_1)\right].$$

which proves (5.1). When c_{11} is large $P\left[d_i \leq \frac{\varepsilon}{k}\right]$ will for $i \neq 1$ be much smaller then $\frac{\varepsilon}{k}$ and therefore in that case Q_1 converges to its upper bound.

The inequalities (5.4) can be derived in the same way.

6. Tables and nomograms

To obtain the values d_i and e_i as defined by (2.7) and (2.9) and to evaluate the power functions (5.4) and (5.5) we need suitable tables or nomograms of the incomplete B function.

When all α_i are equal, the smallest d_i corresponds to the largest ratio x_i and the smallest e_i corresponds to the smallest ratio x_i. Further the critical values $g_{i,\varepsilon}^{(1)}$ of x_i when testing the largest ratio and $g_{i,\varepsilon}^{(2)}$ for testing the smallest ratio are then all equal:

$$g_{i,\varepsilon}^{(1)} = g_{\varepsilon}^{(1)} \quad (i=1,\ldots,k),$$

$$g_{i,\varepsilon}^{(2)} = g_{\varepsilon}^{(2)} \quad (i=1,\ldots,k).$$

Therefore in this case it suffices to have tables with these critical values with entries k and the common parameter value α. These tables may be found in C. EISENHART, M.W. HASTAY and W.A. WALLIS (1947) ($\varepsilon = 0.05$ and 0.01) for the first test and in R. DOORNBOS (1956) ($\varepsilon = 0.05$) for the second one.

When there are unequal values among the α_i the minimum d value may be found in most cases by means of PEARSON's tables of the incomplete B-function (K. PEARSON (1934)).

The smallest e value, however, will, when it lies in the neighbourhood of $\frac{\varepsilon}{k}$ and k is not very small, correspond to such a small ratio x that neither these tables nor the nomograms constructed by H.O. HARTLEY and E.R. FITCH (1951) are suitable for our purpose.

An extension of the tables or nomograms to cover this case seems useful.

7. References

W.G. COCHRAN (1941), The distribution of the largest of a set of estimated variances as a fraction of their total, Annals of Enginics, 11 (1941) pp 47-52.

R. DOORNBOS (1956), Significance of the smallest of a set of estimated normal variances, Statistica Neerlandica, 10 (1956), pp

C. EISENHART, M.W. HASTAY and W.A. WALLIS (1947), Selected Techniques of Statistical Analysis, Chapter 15, New York and London, 1947.

W. FELLER (1950), An Introduction to Probability Theory and its Applications, Volume I, New York and London, 1950.

H.O. HARTLEY and E.R. FITCH (1951), A chart for the incomplete Bêta-function and the cumulative binomial distribution, Biometrika, 38 (1951), pp 423-426.

K. PEARSON (1934), Tables of the incomplete Bêta-function, Cambridge, 1934.